Annotated Examples 8

Greuel, Laplagne, Seelich

This seems to be on page 95 of Vasconcelas (unhomogenized), *Computational Methods in Commutative Algebra and Algebraic Geometry* with no attribution. It is example I_7 from Greuel, Laplagne, and Seelich, given by (homogenized) generators

$$x^2 + zw,\ y^3 + xwt,\ xw^3 + z^3t + ywt^2,\ y^2w^4 - xy^2z^2t - w^3t^3$$

This is clearly homogeneous, maybe not immediately clear that it is a curve. Dehomogenize it by using $Z_{21} := z/t$, $X_{18} := x/t$, $W_{15} := w/t$, $Y_{11} := y/t$, and look at the divisors:

$$div(Z_{21}) = -21 \cdot P_\infty + 12 \cdot Q + 9 \cdot R$$

$$div(X_{18}) = -18 \cdot P_\infty + 5 \cdot Q + 13 \cdot R$$

$$div(W_{15}) = -15 \cdot P_\infty - 2 \cdot Q + 17 \cdot R$$

$$div(Y_{11}) = -11 \cdot P_\infty + 1 \cdot Q + 10 \cdot R$$

Maybe it is no surprise that there exists V_7 with

$$div(V_7) = -7 \cdot P_\infty + 4 \cdot Q + 3 \cdot R$$

and $Z_{21} = V_7^3$, $X_{18} = Y_{11}V_7$. The one-point version would use $U_{37} := W_{15}Y_{11}^2$ in place of W_{15}.

and a Gröbner basis for an almost module presentation is:

$$\begin{gathered}
U_{37}^2 - Y_{11} * f_7^9 - U_{37},\\
U_{37}f_{27} - Y_{11}^2 * f_7^6 - f_{27},\\
Y_{11}^3 * f_7^3 - U_{37} * f_{17} + f_{17},\\
f_{27}^2 - U_{37} * f_{17},\\
U_{37} * Y_{11} - f_{27} * f_7^3,\\
Y_{11}^4 - U_{37} * f_7 + f_7,\\
f_{27} * f_{17} - U_{37} * f_7,\\
f_{27}Y_{11} - f_{17} * f_7^3,\\
f_{17}^2 - f_{27} * f_7,\\
f_{17}Y_{11} - f_7^4.
\end{gathered}$$

If one is then wedded to the original "two-point" presentation, replace U_{37} by $W_{15}Y_{11}^2$ and then magically produce some more elements (see below).

A straigh-forward attempt at this in SINGULAR 3-1-0 gives:

```
                     SINGULAR                             /
 A Computer Algebra System for Polynomial Computations   /   version 3-1-0
                                                        0<
     by: G.-M. Greuel, G. Pfister, H. Schoenemann        \   Mar 2009
FB Mathematik der Universitaet, D-67653 Kaiserslautern    \
> LIB "normal.lib";
> ring r=0,(u,z,x,w,y),wp(37,21,18,15,11);
> ideal i=x2+zw,y3+xw,xw3+z3+yw,y2w4-xy2z2-w3;
> list nor=normal(i);
> nor;
[1]:
   [1]:
//   characteristic : 0
//   number of vars : 15
//        block   1 : ordering dp
//                  : names    T(1) T(2) T(3) T(4) T(5) T(6) T(7) T(8) T(9) T(10)
//        block   2 : ordering wp
//                  : names    u z x w y
//                  : weights  37 21 18 15 11
//        block   3 : ordering C
[2]:
   [1]:
      _[1]=w2
      _[2]=xw
      _[3]=wy2
      _[4]=xy2
      _[5]=zwy
      _[6]=zxy
      _[7]=z2w
      _[8]=z2x
      _[9]=z3+wy
      _[10]=z2y2
      _[11]=y3
> def R=nor[1][1];
> setring R;
> normap;
normap[1]=u
normap[2]=z
normap[3]=x
normap[4]=w
normap[5]=y
> norid;
norid[1]=T(2)+1
norid[2]=-T(3)*y+w
norid[3]=T(8)*w+T(10)*y
norid[4]=-T(4)*y+x
```

```
norid[5]=T(1)*x+T(3)*y
norid[6]=T(3)*x-T(4)*w
norid[7]=T(4)*x+T(5)*y
norid[8]=T(5)*x-T(6)*w
norid[9]=T(6)*x+T(7)*y
norid[10]=T(7)*x+T(10)*y
norid[11]=-T(1)*y+T(8)*x+T(9)*w
norid[12]=T(1)*z-T(4)*y
norid[13]=T(3)*z-T(5)*y
norid[14]=T(4)*z-T(6)*y
norid[15]=T(5)*z-T(7)*y
norid[16]=T(6)*z-T(8)*y
norid[17]=T(1)*y+T(7)*z-T(9)*w
norid[18]=T(8)*z-T(9)*x-y
norid[19]=T(4)*w+y^2
norid[20]=T(1)*y^2-T(3)*w
norid[21]=T(4)*y^2-T(5)*w
norid[22]=T(6)*y^2-T(7)*w
norid[23]=T(7)*y^2-T(10)*w
norid[24]=T(8)*y^2-T(10)*x
norid[25]=-T(3)*y+T(9)*y^2-T(10)*z
norid[26]=T(3)*w*y-T(9)
norid[27]=T(6)*w+z*y
norid[28]=-T(1)*w*y+T(9)*w^2-T(10)*x*y
norid[29]=T(3)*y^2+T(9)*x*w+T(10)*z*y
norid[30]=-T(10)*y+z^2
norid[31]=T(5)*w*y^2-T(9)*z
norid[32]=T(4)*w^2*y-T(9)*x
norid[33]=-T(9)*z^2+T(10)*w^2*y
norid[34]=-T(1)*w+T(3)*w^3-T(10)*x
norid[35]=T(3)*y+T(4)*w^3+T(10)*z
norid[36]=T(5)*w^3-T(9)*x*y
norid[37]=T(6)*w^3+T(9)*z*y
norid[38]=T(7)*w^3-T(9)*z*x
norid[39]=-T(7)*w^2-T(10)*z^2*x+T(10)*w^4
norid[40]=T(1)^2-T(1)*w^4*y-T(10)*w*y^2-T(10)
norid[41]=T(1)*T(2)+T(1)
norid[42]=T(2)^2-1
norid[43]=T(1)*T(3)-T(1)*w^2*y+T(8)*y
norid[44]=T(2)*T(3)+T(3)
norid[45]=T(3)^2-T(1)*y
norid[46]=T(1)*T(4)+T(3)
norid[47]=T(2)*T(4)+T(4)
norid[48]=T(3)*T(4)+y
norid[49]=T(4)^2+T(5)
norid[50]=T(1)*T(5)+y
```

```
norid[51]=T(2)*T(5)+T(5)
norid[52]=T(3)*T(5)-T(1)*z
norid[53]=T(4)*T(5)+z
norid[54]=T(5)^2-T(4)*z
norid[55]=T(1)*T(6)+T(5)
norid[56]=T(2)*T(6)+T(6)
norid[57]=T(3)*T(6)+z
norid[58]=T(4)*T(6)+T(7)
norid[59]=T(5)*T(6)+T(10)
norid[60]=T(6)^2+T(1)*w*y^2-T(1)
norid[61]=T(1)*T(7)+z
norid[62]=T(2)*T(7)+T(7)
norid[63]=T(3)*T(7)-T(4)*z
norid[64]=T(4)*T(7)+T(10)
norid[65]=T(5)*T(7)-T(6)*z
norid[66]=T(6)*T(7)-T(1)*x*w*y-T(3)
norid[67]=T(7)^2-T(1)*z*w^2-y
norid[68]=T(1)*T(8)+T(7)
norid[69]=T(2)*T(8)+T(8)
norid[70]=T(3)*T(8)+T(10)
norid[71]=T(4)*T(8)+T(1)*w*y^2-T(1)
norid[72]=T(5)*T(8)-T(1)*x*w*y-T(3)
norid[73]=T(6)*T(8)+T(1)*z*w*y-T(4)
norid[74]=T(7)*T(8)-T(1)*z*x*w-T(5)
norid[75]=T(8)^2+T(1)*z^2*w-T(6)
norid[76]=T(1)*T(9)-T(1)*w^2
norid[77]=T(2)*T(9)-T(1)*x*w
norid[78]=T(3)*T(9)-T(1)*w*y^2
norid[79]=T(4)*T(9)-T(1)*x*y^2
norid[80]=T(5)*T(9)-T(1)*z*w*y
norid[81]=T(6)*T(9)-T(1)*z*x*y
norid[82]=T(7)*T(9)-T(1)*z^2*w
norid[83]=T(8)*T(9)-T(1)*z^2*x
norid[84]=T(9)^2-T(1)*z^3-T(1)*w*y
norid[85]=T(1)*T(10)-T(4)*z
norid[86]=T(2)*T(10)+T(10)
norid[87]=T(3)*T(10)-T(5)*z
norid[88]=T(4)*T(10)-T(6)*z
norid[89]=T(5)*T(10)-T(7)*z
norid[90]=T(6)*T(10)-T(1)*z*w^2-y
norid[91]=T(7)*T(10)-T(1)*z*w*y^2+T(4)*y
norid[92]=T(8)*T(10)-T(1)*z*x*y^2-z
norid[93]=T(9)*T(10)-T(1)*z^2*y^2
norid[94]=T(10)^2+T(1)*z*x*w*y+T(3)*z
norid[95]=x^2+z*w
norid[96]=x*w+y^3
```

```
norid[97]=z^3+x*w^3+w*y
norid[98]=-z^3*x+z*w^4-x*w*y
norid[99]=-z^2*x*y^2+w^4*y^2-w^3
> option(redSB);
> ideal j=std(norid);j;
j[1]=x*w+y^3
j[2]=x^2+z*w
j[3]=z*w^2-x*y^3
j[4]=z^3-w^2*y^3+w*y
j[5]=z^2*x*y^2-w^4*y^2+w^3
j[6]=w^5*y^2+z^2*y^5-w^4
j[7]=T(10)*y-z^2
j[8]=T(10)*z-w^2*y^2+w
j[9]=T(10)*w^2-z*x*y^2
j[10]=T(9)-w^2
j[11]=T(8)*w+z^2
j[12]=T(8)*z+w*y^3-y
j[13]=T(8)*y^2-T(10)*x
j[14]=T(7)*w-z*x
j[15]=T(7)*x+z^2
j[16]=T(7)*z+T(8)*x
j[17]=T(7)*y^2-T(10)*w
j[18]=T(6)*w+z*y
j[19]=T(6)*x+T(7)*y
j[20]=T(6)*z-T(8)*y
j[21]=T(6)*y^2-z*x
j[22]=T(5)*w-x*y
j[23]=T(5)*x+z*y
j[24]=T(5)*z-T(7)*y
j[25]=T(5)*y^2-z*w
j[26]=T(4)*y-x
j[27]=T(4)*w+y^2
j[28]=T(4)*x+T(5)*y
j[29]=T(4)*z-T(6)*y
j[30]=T(3)*y-w
j[31]=T(3)*w-T(8)*x*y-w^3*y
j[32]=T(3)*x+y^2
j[33]=T(3)*z-T(5)*y
j[34]=T(2)+1
j[35]=T(1)*y-T(8)*x-w^3
j[36]=T(1)*w+T(10)*x-w^5*y-z^2*y^4
j[37]=T(1)*x+w
j[38]=T(1)*z-x
j[39]=T(10)^2+T(5)*y-x*y^4
j[40]=T(8)*T(10)+z*w*y^2-z
j[41]=T(7)*T(10)+y^5+x
```

```
j[42]=T(6)*T(10)+w*y^3-y
j[43]=T(5)*T(10)+T(8)*x
j[44]=T(4)*T(10)-T(8)*y
j[45]=T(3)*T(10)-T(7)*y
j[46]=T(1)*T(10)-T(6)*y
j[47]=T(8)^2-T(6)-z*y^3
j[48]=T(7)*T(8)-T(5)+x*y^3
j[49]=T(6)*T(8)-T(4)-y^4
j[50]=T(5)*T(8)-T(3)+w^2*y
j[51]=T(4)*T(8)-T(1)-z^2*x*y+w^4*y
j[52]=T(3)*T(8)+T(10)
j[53]=T(1)*T(8)+T(7)
j[54]=T(7)^2+w*y^3-y
j[55]=T(6)*T(7)-T(3)+w^2*y
j[56]=T(5)*T(7)-T(8)*y
j[57]=T(4)*T(7)+T(10)
j[58]=T(3)*T(7)-T(6)*y
j[59]=T(1)*T(7)+z
j[60]=T(6)^2-T(1)-z^2*x*y+w^4*y
j[61]=T(5)*T(6)+T(10)
j[62]=T(4)*T(6)+T(7)
j[63]=T(3)*T(6)+z
j[64]=T(1)*T(6)+T(5)
j[65]=T(5)^2-T(6)*y
j[66]=T(4)*T(5)+z
j[67]=T(3)*T(5)-x
j[68]=T(1)*T(5)+y
j[69]=T(4)^2+T(5)
j[70]=T(3)*T(4)+y
j[71]=T(1)*T(4)+T(3)
j[72]=T(3)^2-T(8)*x-w^3
j[73]=T(1)*T(3)+T(8)*y-w^5-z^2*y^3
j[74]=T(1)^2-T(10)-w^7-z*x*y^6-z^2*w*y
```

Now try the same with V_7:

```
> ring r=0,(w,y,u),wp(15,11,7);
> ideal i=y2+uw,yuw3+u9+yw,y2w4-y3u7-w3;
> list nor=normal(i);
> nor;
[1]:
   [1]:
//   characteristic : 0
//   number of vars : 3
//        block   1 : ordering wp
//                  : names    w y u
//                  : weights  15 11 7
```

```
//        block   2 : ordering C
   [2]:
//   characteristic : 0
//   number of vars : 8
//        block   1 : ordering dp
//                  : names    T(1) T(2) T(3) T(4) T(5)
//        block   2 : ordering wp
//                  : names    w y u
//                  : weights  15 11 7
//        block   3 : ordering C
[2]:
   [1]:
      _[1]=1
   [2]:
      _[1]=-w3y-u8
      _[2]=-w4+yu7
      _[3]=wu
      _[4]=yu4
      _[5]=u7
      _[6]=-wu
> def R=nor[1][1];
> setring R;
> normap;
normap[1]=w
normap[2]=y
normap[3]=u
> norid;
norid[1]=u
norid[2]=y
norid[3]=w
> setring r;
> def R2=nor[1][2];
> setring R2;
> normap;
normap[1]=w
normap[2]=y
normap[3]=u
> norid;
norid[1]=T(3)+1
norid[2]=T(1)*y+T(2)*u
norid[3]=T(1)*w-T(2)*y
norid[4]=T(1)*u^2+y
norid[5]=T(2)*u^2+w
norid[6]=T(4)*u^3-T(5)*y
norid[7]=T(4)*y*u^2+T(5)*w
norid[8]=T(1)+T(2)*w*y*u+T(5)*u
```

```
norid[9]=T(2)*w^2*u+T(2)-T(5)*y
norid[10]=-T(1)*u-T(5)*u^2+w^2*y
norid[11]=-T(2)*u+T(5)*y*u+w^3
norid[12]=-T(4)*y+u^4
norid[13]=T(4)*w+y*u^3
norid[14]=T(4)*w^3+T(5)*u^5-y*u^2
norid[15]=T(1)^2+T(2)*w^2-T(4)*u^2
norid[16]=T(1)*T(2)+T(1)*w^5+T(2)*w*u^8+u^5
norid[17]=T(2)^2-T(1)*w^2*u^7+T(1)*u^6+T(2)*w^5
norid[18]=T(1)*T(3)+T(1)
norid[19]=T(2)*T(3)+T(2)
norid[20]=T(3)^2-1
norid[21]=T(1)*T(4)+u^2
norid[22]=T(2)*T(4)+T(1)*u^3
norid[23]=T(3)*T(4)+T(4)
norid[24]=T(4)^2+T(1)*w^2*u+T(1)
norid[25]=T(1)*T(5)+T(4)*u
norid[26]=T(2)*T(5)-u^4
norid[27]=T(3)*T(5)+T(5)
norid[28]=T(4)*T(5)+T(2)*w*u^4-u
norid[29]=T(5)^2-T(1)*w*u^6-T(4)
norid[30]=y^2+w*u
norid[31]=w^3*y*u+u^9+w*y
norid[32]=-w^4*u+y*u^8-w^2
> option(redSB);
> ideal j=std(norid);j;
j[1]=y^2+w*u
j[2]=w^3*y*u+u^9+w*y
j[3]=w^4*u-y*u^8+w^2
j[4]=T(5)*w+u^6
j[5]=T(5)*u^3-w^2*y*u-y
j[6]=T(5)*y*u^2+w^3*u+w
j[7]=T(4)*y-u^4
j[8]=T(4)*w+y*u^3
j[9]=T(4)*u^3-T(5)*y
j[10]=T(3)+1
j[11]=T(2)-T(5)*y+w^5-w*y*u^7
j[12]=T(1)+T(5)*u+w^4*y+w*u^8
j[13]=T(5)^2-T(4)+w*y*u^4
j[14]=T(4)*T(5)-w^2*u^2-u
j[15]=T(4)^2-T(5)*u
```

Ignoring the silly first ring, we see that there are three unnecessary variables $T(1) = -T(5)u - w^4y - wu^8$, $T(2) = T(5)y - w^5 + wyu^7$, and $T(3) = -1$. So

```
> ring r1=0,(f27,f17,f15,f11,f7),wp(27,17,15,11,7);
> map phi=R2,-f27*f7-f15^4*f11-f15*f7^8,f27*f11-f15^5+f15*f11*f7^7,-1,f17,f27,f15,f11,f7
```

```
> ideal i1=phi(j);
> option(redSB);
> ideal j1=std(i1);j1;
j1[1]=f11^2+f15*f7
j1[2]=f17*f11-f7^4
j1[3]=f17*f15+f11*f7^3
j1[4]=f17^2-f27*f7
j1[5]=f27*f11-f17*f7^3
j1[6]=f27*f15+f7^6
j1[7]=f27*f17-f15^2*f7^2-f7
j1[8]=f15^2*f11*f7-f27*f7^3+f11
j1[9]=f15^3*f7+f17*f7^5+f15
j1[10]=f27^2+f15*f11*f7^4-f17
```

or

```
> intmat A[5][5]=1,1,1,1,0,1,1,1,0,0,1,1,0,0,0,1,0,0,0,0,27,17,15,11,7;
> ring r1=0,(f27,f17,f15,f11,f7),M(A);
> map phi=R2,-f27*f7-f15^4*f11-f15*f7^8,f27*f11-f15^5+f15*f11*f7^7,-1,f17,f27,f15,f11,f7
> ideal i1=phi(j);
> option(redSB);
> ideal j1=std(i1);j1;
j1[1]=f11^2+f15*f7
j1[2]=f17*f11-f7^4
j1[3]=f27*f11-f17*f7^3
j1[4]=f17*f15+f11*f7^3
j1[5]=f27*f15+f7^6
j1[6]=f17^2-f27*f7
j1[7]=f27*f17-f15^2*f7^2-f7
j1[8]=f27^2+f15*f11*f7^4-f17
j1[9]=f15^2*f11*f7-f27*f7^3+f11
j1[10]=f15^3*f7+f17*f7^5+f15
\end[verbatim}
```

take your pick.

Let's try Macaulay2:

```
\begin{verbatim}
Macaulay 2, version 1.2
with packages: Elimination, IntegralClosure, LLLBases, PrimaryDecomposition,
               ReesAlgebra, SchurRings, TangentCone

i1 : R=QQ[z,x,w,y]

o1 = R
```

```
o1 : PolynomialRing

i2 : I=ideal (x^2+z*w,y^3+x*w,x*w^3+z^3+y*w,y^2*w^4-x*y^2*z^2-w^3)

              2         3            3    3         4 2    2   2    3
o2 = ideal (x  + z*w, y  + x*w, x*w  + z  + w*y, w y  - z x*y  - w )

o2 : Ideal of R

i3 : S=R/I

o3 = S

o3 : QuotientRing

i4 : F=icFractions(S)

                                3
            2    2       w  - 99w   -w    2     4    2       2      2
      -w  -y   -y   z*x   3           5   y   - w  + z x  -z*y   -z*y
o4 = {--, ---, ---, ---, ---------, ---, --, ----------, -----, -----,
       x   x    w    w       x        y   w        w        x      w
                                                3
     ---------------------------------------------------------------------------
     - w  + w y - z x*w*y  - w  + 11w y - 11z x*w*y             2      2
        0                     0                       -z*x*y  -11w   -w*y
     --------------------, -----------------------, ------, -----, -----,
               x                      y                w     11x     x
     ---------------------------------------------------------------------------
     z*x*y
     ------, z, x, w, y}
        w

o4 : List

i5 : P=presentation(integralClosure(S))

o5 = | 10w_9^11w_11^3w_14^4+10w_9^3w_11^2w_14^3+10w_9^5w_14^2

       -w_9^2w_11^2w_14^2-w_9^3w_14^3 -w_9^3w_11^2w_14^2-w_9^4w_14^3

       w_9^3w_11^2w_14^2+w_9^4w_14^3 w_9^4w_11^2w_14^2+w_9^5w_14^3

       -w_9^9w_11^3w_14^3+w_9^2w_14^3-w_9^3w_14 w_9^2w_11^2w_14^2+w_9^3w_14^3
```

```
       -w_9^8w_11^3w_14^3+w_9w_14^3-w_9^2w_14

       -w_9^2w_11w_13w_14+99w_9^2w_14^3-100w_9^3w_14

       -w_9w_11w_13w_14+99w_9w_14^3-100w_9^2w_14

       w_9^2w_11w_13w_14^2-99w_9^2w_14^4+100w_9^3w_14^2

       -w_9w_11^2w_14^2-w_9^2w_14^3

       w_9^2w_13w_14^2+99w_9w_11w_14^3-100w_9^2w_11w_14

       w_9^3w_11^2w_14+w_9^4w_14^2 -w_9^2w_11^2w_14-w_9^3w_14^2

       -w_9w_11^2w_14-w_9^2w_14^2 w_9^9w_11^3w_14^3+w_9w_11^2w_14^2+w_9^3w_14

       w_9^3w_11^2w_14+w_9^4w_14^2 w_9^2w_11^2w_14+w_9^3w_14^2

       100w_9^10w_11^3w_14^3-w_9^3w_11w_13w_14+w_9^2w_11^2w_14^2

       -99w_9^8w_11^2w_14^3+w_9w_13w_14-w_9w_11 w_9^8w_11w_14^3+w_14^2-w_9

       -99w_9^7w_11^2w_14^3+w_13w_14-w_11 w_9^8w_11^2w_14^3+w_11w_14^2-w_9w_11

       10000w_9^27w_11^8w_14^9-10000w_9^11w_11^6w_14^7-20000w_9^13w_11^4w_14^6-

       199w_9^6w_11^3w_14^3+w_13^2+w_14 -99w_9^7w_11^3w_14^3+w_11w_13w_14+w_9w_14

       99w_9^8w_11w_14^3+w_11w_13+w_9 w_11^2w_14+w_9w_14^2

       -w_9w_11^2w_14-w_9^2w_14^2 w_11^2+w_9w_14 -w_9w_11^2-w_9^2w_14

       w_9^2w_11^2+w_9^3w_14

                                     1                             32
o5 : Matrix (QQ[w , w  , w  , w  ])  <--- (QQ[w , w  , w  , w  ])
                 9   11   13   14              9   11   13   14

i6 : g=gens gb ideal(P)

o6 = | w_11w_13-99w_14^2+100w_9
       w_11^2+w_9w_14
       w_9w_13w_14+99w_11w_14^2-100w_9w_11
       9801w_9^7w_11w_14^4-10000w_9^8w_11w_14^2-w_13^2-w_14
       99w_9^8w_14^4+w_13w_14-w_11 w_9^8w_11w_14^3+w_14^2-w_9
       100w_9^9w_14^3+w_9w_13-w_11w_14
```

```
10000w_9^9w_11w_14^2+w_9w_13^2+9801w_14^3-9800w_9w_14
1000000w_9^10w_14^2-w_9w_13^3-9801w_13w_14^3-1960200w_11w_14^2+1970000w_9w_11
100000000w_9^10w_11w_14-w_9w_13^4-9801w_13^2w_14^3-194059800w_14^4+490050000w_9w_
10000000000w_9^11w_14+w_9w_13^5+9801w_13^3w_14^3+194059800w_13w_14^4+48514950000w_
1000000000000w_9^11w_11+w_9w_13^6+9801w_13^4w_14^3+194059800w_13^2w_14^4+
 4802980050000w_14^5+494000000w_9^2w_13^2-8742492000000w_9w_14^3+
 3940200000000w_9^2w_14 100000000000000w_9^12-w_9w_13^7-9801w_13^5w_14^3-
 194059800w_13^3w_14^4-4802980050000w_13w_14^5-593000000w_9^2w_13^3-
 769447107000000w_11w_14^4+973239300000000w_9w_11w_14^2-198990000000000w_9^2w_11 9
96059601w_9^6w_11w_14^7-100000000w_9^9w_11w_14+w_13^4-29799w_13^2w_14+
 1960200w_14^2-1990000w_9
9509900499w_9^6w_14^9+10000000000w_9^10w_14+w_13^5-39699w_13^3w_14+
 198960300w_13w_14^2-1990000w_9w_13-197010000w_11w_14
941480149401w_9^5w_11w_14^10-1000000000000w_9^10w_11-w_13^6+49599w_13^4w_14
93206534790699w_9^5w_14^12+100000000000000w_9^11-w_13^7+59499w_13^5w_14-        8
 3900798000000w_11w_14^2-5910300000000w_9w_11
9227446944279201w_9^4w_11w_14^13-100000000000000w_9^11w_13+w_13^8-69399w_13^6w_14+13
913517247483640899w_9^4w_14^15-100000000000000w_9^11w_13^2+w_13^9-79299w_13^7w_14+19
69394050000000000w_9^2w_13+97908069600000000000w_9w_11w_14
90438207500880449001w_9^3w_11w_14^16+100000000000000w_9^11w_13^3-w_13^10+
 89199w_13^8w_14-2654110800w_13^6w_14^2+298990000w_9w_13^6+32233430790000w_13^4w_
118497060000000000w_9^2w_13^2-280311618708000000000w_9w_14^3+
13652596980000000000w_9^2w_14
8953382542587164451099w_9^3w_14^18+100000000000000w_9^11w_13^4-w_13^11+
99099w_13^9w_14-3439170900w_13^7w_14^2+397990000w_9w_13^7+52677630720000w_13^5w_14^3-165
886384871716129280658801w_9^2w_11w_14^19-100000000000000w_9^11w_13^5+w_13^12-108999
---------------------------------------------------------------------------    877
---------------------------------------------------------------------------    114
---------------------------------------------------------------------------    633
---------------------------------------------------------------------------    452
---------------------------------------------------------------------------    3w_
---------------------------------------------------------------------------    868
---------------------------------------------------------------------------    158
---------------------------------------------------------------------------    120
---------------------------------------------------------------------------    393
---------------------------------------------------------------------------    13^
---------------------------------------------------------------------------    673
---------------------------------------------------------------------------    860
---------------------------------------------------------------------------    15+
---------------------------------------------------------------------------    212
---------------------------------------------------------------------------    226
---------------------------------------------------------------------------    868
---------------------------------------------------------------------------    156
---------------------------------------------------------------------------    596
---------------------------------------------------------------------------    851
---------------------------------------------------------------------------    13^
```

```
     ---------------------------------------------------------------------------     2763
     ---------------------------------------------------------------------------     4051
     ---------------------------------------------------------------------------     7510
     ---------------------------------------------------------------------------     3809
     ---------------------------------------------------------------------------     3000
     ---------------------------------------------------------------------------     5202
     ---------------------------------------------------------------------------     8429
     ---------------------------------------------------------------------------     3521
     ---------------------------------------------------------------------------     6909
     ---------------------------------------------------------------------------     2595
     ---------------------------------------------------------------------------     6820
     ---------------------------------------------------------------------------     6002
     ---------------------------------------------------------------------------     1181
     ---------------------------------------------------------------------------     8238
     ---------------------------------------------------------------------------     4827
     ---------------------------------------------------------------------------     2457

                                   1                             28
o6 : Matrix (QQ[w , w  , w  , w  ])  <--- (QQ[w , w  , w  , w  ])
                 9   11   13   14                9   11   13   14
```

I haven't a clue as to what w_0, w_3, and w_5 are in icFractions; nor do I have a clue as to what w_9, w_{11}, w_{13}, and w_{14} are in the presentation, let alone whether the two answers are related or correct in any sense.

Now consider the fact that there are magical elements $A := Y/V$, $B := A/V$, $C := B/V$ with divisors

$$div(A) = -4 \cdot P_\infty - 3 \cdot Q + 7 \cdot R$$

$$div(B) = 3 \cdot P_\infty - 7 \cdot Q + 4 \cdot R$$

$$div(C) = 10 \cdot P_\infty - 11 \cdot Q + 1 \cdot R$$

which told me they clearly existed and had poles only at P_∞ and Q. So everything in the integral closure can be written in terms of V and C with

$$C^7V^{11} + C^3 - V = 0,$$

namely, $Y = CV^3$, $W = -C^2V^5$, $A = CV^2$, $B = CV$, (and $Z = V^3$, $X = YV = CV^4$).

Let's analyze the answer in SINGULAR 3-1-0:

```
                     SINGULAR                             /
 A Computer Algebra System for Polynomial Computations   /   version 3-1-0
                                                       0<
     by: G.-M. Greuel, G. Pfister, H. Schoenemann        \   Mar 2009
FB Mathematik der Universitaet, D-67653 Kaiserslautern    \
> LIB "/home/leonada/presolve.lib";
> LIB "/home/leonada/normal.lib";
> ring r=0,(w,y,u),wp(15,11,7);
> ideal i=y7+y3u-u11,y2+wu;
> list nor=normal(i);
> nor;
[1]:
   [1]:
//   characteristic : 0
//   number of vars : 8
//        block   1 : ordering dp
//                  : names    T(1) T(2) T(3) T(4) T(5)
//        block   2 : ordering wp
//                  : names    w y u
//                  : weights  15 11 7
//        block   3 : ordering C
   [2]:
      //   characteristic : 0
//   number of vars : 3
//        block   1 : ordering wp
//                  : names    w y u
//                  : weights  15 11 7
```

```
//        block   2 : ordering C
[2]:
   [1]:
      _[1]=-w3y-u8
      _[2]=-w4+yu7
      _[3]=wu
      _[4]=yu4
      _[5]=u7
      _[6]=-wu
   [2]:
      _[1]=1
> def R=nor[1][1];
> setring R;
> normap;
normap[1]=w
normap[2]=y
normap[3]=u
> norid;
norid[1]=T(3)+1
norid[2]=T(1)*y+T(2)*u
norid[3]=T(1)*w-T(2)*y
norid[4]=T(1)*u^2+y
norid[5]=T(2)*u^2+w
norid[6]=T(4)*u^3-T(5)*y
norid[7]=T(4)*y*u^2+T(5)*w
norid[8]=T(1)+T(2)*w*y*u+T(5)*u
norid[9]=T(2)*w^2*u+T(2)-T(5)*y
norid[10]=-T(1)*u-T(5)*u^2+w^2*y
norid[11]=-T(2)*u+T(5)*y*u+w^3
norid[12]=-T(4)*y+u^4
norid[13]=T(4)*w+y*u^3
norid[14]=T(4)*w^3+T(5)*u^5-y*u^2
norid[15]=T(1)^2+T(2)*w^2-T(4)*u^2
norid[16]=T(1)*T(2)+T(1)*w^5+T(2)*w*u^8+u^5
norid[17]=T(2)^2-T(1)*w^2*u^7+T(1)*u^6+T(2)*w^5
norid[18]=T(1)*T(3)+T(1)
norid[19]=T(2)*T(3)+T(2)
norid[20]=T(3)^2-1
norid[21]=T(1)*T(4)+u^2
norid[22]=T(2)*T(4)+T(1)*u^3
norid[23]=T(3)*T(4)+T(4)
norid[24]=T(4)^2+T(1)*w^2*u+T(1)
norid[25]=T(1)*T(5)+T(4)*u
norid[26]=T(2)*T(5)-u^4
norid[27]=T(3)*T(5)+T(5)
norid[28]=T(4)*T(5)+T(2)*w*u^4-u
```

```
norid[29]=T(5)^2-T(1)*w*u^6-T(4)
norid[30]=y^2+w*u
norid[31]=w^3*y*u+u^9+w*y
norid[32]=-w^4*u+y*u^8-w^2
> option(redSB);
> ideal j=std(norid);j;
j[1]=y^2+w*u
j[2]=w^3*y*u+u^9+w*y
j[3]=w^4*u-y*u^8+w^2
j[4]=T(5)*w+u^6
j[5]=T(5)*u^3-w^2*y*u-y
j[6]=T(5)*y*u^2+w^3*u+w
j[7]=T(4)*y-u^4
j[8]=T(4)*w+y*u^3
j[9]=T(4)*u^3-T(5)*y
j[10]=T(3)+1
j[11]=T(2)-T(5)*y+w^5-w*y*u^7
j[12]=T(1)+T(5)*u+w^4*y+w*u^8
j[13]=T(5)^2-T(4)+w*y*u^4
j[14]=T(4)*T(5)-w^2*u^2-u
j[15]=T(4)^2-T(5)*u
```

It is not too hard to figure out that $T(3) = -1$ $T(2) = T(5)y - w^5 + wyu^7$, and $T(1) = -T(5)u - w^4y - wu^8$ may not be necessary. $T(4) = -yu^3/w = V/C$, $T(5) = -u^6/w = V/C^2$. So can C be recovered from this answer? I'm guessing not; at least the following is the best I can do.

```
> ring r1=0,(c,T(4),T(5),w,y,u),lp;
> map phi=R,-T(5)*u-w^4*y-w*u^8,T(5)*y-w^5+w*y*u^7,-1,T(4),T(5),w,y,u;
> ideal i1=phi(j);
> ideal j1=i1,c^7*u^11+c^3-u;
> option(redSB);
> ideal k1=std(j1);k1;
k1[1]=y^7+y^3*u-u^11
k1[2]=w*u+y^2
k1[3]=w*y^5-y^3+u^10
k1[4]=w^2*y^3-w*y-u^9
k1[5]=w^3*y^2-w^2+y*u^8
k1[6]=T(5)*u^3+w*y^3-y
k1[7]=T(5)*y*u^2-w^2*y^2+w
k1[8]=T(5)*y^2-u^7
k1[9]=T(5)*w+u^6
k1[10]=T(5)^2*y-y^4*u^3-u^4
k1[11]=T(5)^3-y^4-y*u^10-u
k1[12]=T(4)-T(5)^2+y^3*u^3
k1[13]=c^7*u^11+c^3-u
```

```
k1[14]=c^7*y*u^8+c^3*T(5)-c^3*w^6*y+c^3*y^6*u^5-c^3*y^2*u^6-T(5)*u
       -w^4*y+y^2*u^7
k1[15]=c^7*y^2*u^5+c^3*T(5)^2+c^3*w^11-3*c^3*y^4*u^12-5*c^3*y^3*u^3
       +c^3*y*u^22+3*c^3*u^13-T(5)^2*u+w^9+y^4*u^13+4*y^3*u^4-2*u^14
k1[16]=c^7*y^3-c^7*u^10-c^3*w^21*y-9*c^3*w^2-35*c^3*y^6*u^16
       -28*c^3*y^5*u^7+6*c^3*y^4*u^35+35*c^3*y^3*u^26+21*c^3*y^2*u^17
       -c^3*y*u^45+8*c^3*y*u^8-15*c^3*u^36-w^19*y-8*w*y^2+20*y^6*u^17
       +21*y^5*u^8-y^4*u^36-15*y^3*u^27-15*y^2*u^18-7*y*u^9+5*u^37
k1[17]=c^7*w*y+c^7*u^9-c^3*w^23*y-10*c^3*w^4+56*c^3*y^6*u^15
       -c^3*y^5*u^43+36*c^3*y^5*u^6-21*c^3*y^4*u^34-70*c^3*y^3*u^25
       -28*c^3*y^2*u^16+6*c^3*y*u^44+c^3*y*u^7+35*c^3*u^35-w^21*y-9*w^2
       -35*y^6*u^16-28*y^5*u^7+6*y^4*u^35+35*y^3*u^26+21*y^2*u^17
       -y*u^45+8*y*u^8-15*u^36
k1[18]=c^7*w^2+c^3*T(5)-c^3*w^24-11*c^3*w^6*y+c^3*y^6*u^42
       -25*c^3*y^6*u^5+21*c^3*y^5*u^33+70*c^3*y^4*u^24+84*c^3*y^3*u^15
       -6*c^3*y^2*u^43-2*c^3*y^2*u^6-35*c^3*y*u^34-56*c^3*u^25-T(5)*u
       -w^22-10*w^4*y+28*y^6*u^6-6*y^5*u^34-35*y^4*u^25-56*y^3*u^16
       +y^2*u^44+2*y^2*u^7+15*y*u^35+35*u^26
```

Versions of icFractions and integralClosure in Macaulay2 version 1.2 don't do well; but the icFractions in Macaulay2 version 1.2.1 is supposed to produce $a := x/y = V$,$b := -z^2/w = V/C^2$, and $c := -w/x = CV$. At least here I think I can recover $V = a$ and $C = b(1 - c^7a^3)$, though I need to recheck this.

With MAGMA's Normalisation (and Ideal ?), it is possible to reverse the order of the variables and weight them to get a fairly quick reasonable answer to the type I subproblem: :

```
t:=Cputime();
P<y,x> := PolynomialRing(RationalField(),2);
I := Ideal(y^7+y^3*x-x^11);
Js:=Normalisation(I);
N := Js[1][1];
N<[a]> := N;
N;
G:=GroebnerBasis(N);G;
Cputime(t);
tt:=Cputime();
R<f37,f27,f17,f11,f7>:=PolynomialRing(Rationals(),5,"grevlexw",[37,27,17,11,7]);
hNR:=hom<N->R|R.5,R.4,R.3,R.2,R.1>;
GR:=[G[i]@hNR: i in [1..#G]];
IR:=ideal<R|GR>;
BR:=GroebnerBasis(IR);BR;
Cputime(tt);
```

```
Loading file "11_7a"
Ideal of Polynomial ring of rank 5 over Rational Field
Order: Lexicographical
Variables: a[1], a[2], a[3], a[4], a[5]
Basis:
[
    -a[1]*a[5] + a[2]^4,
    -a[1]^3*a[4] + a[2]*a[5] + a[2],
    -a[1]^3*a[3] + a[2]*a[4],
    -a[1]^4 + a[2]*a[3],
    -a[1]^4*a[4] + a[1]*a[2] + a[2]^5,
    -a[1]^6*a[3] + a[2]^2*a[5] + a[2]^2,
    -a[1]^7 + a[2]^2*a[4],
    -a[1]^4*a[2] + a[2]^2*a[3],
    -a[1]^7*a[3] + a[1]*a[2]^2 + a[2]^6,
    -a[1]^10 + a[2]^3*a[5] + a[2]^3,
    -a[1]^7*a[2] + a[2]^3*a[4],
    -a[1]^4*a[2]^2 + a[2]^3*a[3],
    -a[1]^9*a[2] + a[5]^2 + a[5],
    -a[1]^6*a[2]^2 + a[4]*a[5],
    -a[1]^3*a[2]^3 + a[3]*a[5],
    -a[1]^3*a[2]^3 - a[3] + a[4]^2,
    -a[1]*a[5] - a[1] + a[3]*a[4],
    -a[1]*a[4] + a[3]^2
```

```
]
[
    a[1] + a[2]^4 - a[3]*a[4],
    a[2]^15 - a[3] - a[4]^2*a[5]^3 + a[4]^2*a[5]^2 - a[4]^2*a[5] + a[4]^2,
    a[2]^11*a[3] - a[4]*a[5]^3,
    a[2]^7*a[3]^2 - a[5]^3 - a[5]^2,
    a[2]^4*a[4] + a[3]^2 - a[3]*a[4]^2,
    a[2]^4*a[5] + a[2]^4 + a[3]*a[4] - a[4]^3,
    a[2]^3*a[3]^4 - a[4]*a[5]^2 - a[4]*a[5],
    a[2]^3*a[3]^3*a[4] - a[5]^3 - 2*a[5]^2 - a[5],
    a[2]^3*a[3]^2*a[4]^3 - a[5]^4 - 3*a[5]^3 - 3*a[5]^2 - a[5],
    a[2]^2*a[3]^9 - a[4]*a[5]^4 - 3*a[4]*a[5]^3 - 3*a[4]*a[5]^2 - a[4]*a[5],
    a[2]^2*a[3]^8*a[4] - a[5]^5 - 4*a[5]^4 - 6*a[5]^3 - 4*a[5]^2 - a[5],
    a[2]^2*a[3]^7*a[4]^3 - a[5]^6 - 5*a[5]^5 - 10*a[5]^4 - 10*a[5]^3 - 5*a[5]^2
        - a[5],
    a[2]*a[3]^14 - a[4]*a[5]^6 - 5*a[4]*a[5]^5 - 10*a[4]*a[5]^4 - 10*a[4]*a[5]^3       -
    a[2]*a[3]^13*a[4] - a[5]^7 - 6*a[5]^6 - 15*a[5]^5 - 20*a[5]^4 - 15*a[5]^3 -
        6*a[5]^2 - a[5],
    a[2]*a[3]^12*a[4]^3 - a[5]^8 - 7*a[5]^7 - 21*a[5]^6 - 35*a[5]^5 - 35*a[5]^4
        - 21*a[5]^3 - 7*a[5]^2 - a[5],
    a[2]*a[4]^4 - a[3]^7,
    a[2]*a[4]^2*a[5] + a[2]*a[4]^2 - a[3]^6,
    a[2]*a[5]^2 + 2*a[2]*a[5] + a[2] - a[3]^5,
    a[3]^19 - a[4]*a[5]^8 - 7*a[4]*a[5]^7 - 21*a[4]*a[5]^6 - 35*a[4]*a[5]^5 -
        35*a[4]*a[5]^4 - 21*a[4]*a[5]^3 - 7*a[4]*a[5]^2 - a[4]*a[5],
    a[3]^18*a[4] - a[5]^9 - 8*a[5]^8 - 28*a[5]^7 - 56*a[5]^6 - 70*a[5]^5 -
        56*a[5]^4 - 28*a[5]^3 - 8*a[5]^2 - a[5],
    a[3]^17*a[4]^3 - a[5]^10 - 9*a[5]^9 - 36*a[5]^8 - 84*a[5]^7 - 126*a[5]^6 -
        126*a[5]^5 - 84*a[5]^4 - 36*a[5]^3 - 9*a[5]^2 - a[5],
    a[3]^16*a[4]^5 - a[5]^11 - 10*a[5]^10 - 45*a[5]^9 - 120*a[5]^8 - 210*a[5]^7
        - 252*a[5]^6 - 210*a[5]^5 - 120*a[5]^4 - 45*a[5]^3 - 10*a[5]^2 - a[5],
    a[3]^15*a[4]^7 - a[5]^12 - 11*a[5]^11 - 55*a[5]^10 - 165*a[5]^9 - 330*a[5]^8       -
        11*a[5]^2 - a[5],
    a[3]^14*a[4]^9 - a[5]^13 - 12*a[5]^12 - 66*a[5]^11 - 220*a[5]^10 -
        495*a[5]^9 - 792*a[5]^8 - 924*a[5]^7 - 792*a[5]^6 - 495*a[5]^5 -
        220*a[5]^4 - 66*a[5]^3 - 12*a[5]^2 - a[5],
    a[3]^13*a[4]^11 - a[5]^14 - 13*a[5]^13 - 78*a[5]^12 - 286*a[5]^11 -
        715*a[5]^10 - 1287*a[5]^9 - 1716*a[5]^8 - 1716*a[5]^7 - 1287*a[5]^6 -
        715*a[5]^5 - 286*a[5]^4 - 78*a[5]^3 - 13*a[5]^2 - a[5],
    a[3]^12*a[4]^13 - a[5]^15 - 14*a[5]^14 - 91*a[5]^13 - 364*a[5]^12 -
        1001*a[5]^11 - 2002*a[5]^10 - 3003*a[5]^9 - 3432*a[5]^8 - 3003*a[5]^7 -
        2002*a[5]^6 - 1001*a[5]^5 - 364*a[5]^4 - 91*a[5]^3 - 14*a[5]^2 - a[5],
    a[3]^11*a[4]^15 - a[5]^16 - 15*a[5]^15 - 105*a[5]^14 - 455*a[5]^13 -
        1365*a[5]^12 - 3003*a[5]^11 - 5005*a[5]^10 - 6435*a[5]^9 - 6435*a[5]^8 -       5
        15*a[5]^2 - a[5],
    a[3]^10*a[4]^17 - a[5]^17 - 16*a[5]^16 - 120*a[5]^15 - 560*a[5]^14 -
```

```
    1820*a[5]^13 - 4368*a[5]^12 - 8008*a[5]^11 - 11440*a[5]^10 -
    12870*a[5]^9 - 11440*a[5]^8 - 8008*a[5]^7 - 4368*a[5]^6 - 1820*a[5]^5 -
    560*a[5]^4 - 120*a[5]^3 - 16*a[5]^2 - a[5],
a[3]^9*a[4]^19 - a[5]^18 - 17*a[5]^17 - 136*a[5]^16 - 680*a[5]^15 -
    2380*a[5]^14 - 6188*a[5]^13 - 12376*a[5]^12 - 19448*a[5]^11 -
    24310*a[5]^10 - 24310*a[5]^9 - 19448*a[5]^8 - 12376*a[5]^7 - 6188*a[5]^6
a[3]^8*a[4]^21 - a[5]^19 - 18*a[5]^18 - 153*a[5]^17 - 816*a[5]^16 -
    3060*a[5]^15 - 8568*a[5]^14 - 18564*a[5]^13 - 31824*a[5]^12 -
    43758*a[5]^11 - 48620*a[5]^10 - 43758*a[5]^9 - 31824*a[5]^8 -
    18564*a[5]^7 - 8568*a[5]^6 - 3060*a[5]^5 - 816*a[5]^4 - 153*a[5]^3 -
    18*a[5]^2 - a[5],
a[3]^7*a[4]^23 - a[5]^20 - 19*a[5]^19 - 171*a[5]^18 - 969*a[5]^17 -
    3876*a[5]^16 - 11628*a[5]^15 - 27132*a[5]^14 - 50388*a[5]^13 -
    75582*a[5]^12 - 92378*a[5]^11 - 92378*a[5]^10 - 75582*a[5]^9 -
    50388*a[5]^8 - 27132*a[5]^7 - 11628*a[5]^6 - 3876*a[5]^5 - 969*a[5]^4 -
    171*a[5]^3 - 19*a[5]^2 - a[5],
a[3]^6*a[4]^25 - a[5]^21 - 20*a[5]^20 - 190*a[5]^19 - 1140*a[5]^18 -
    4845*a[5]^17 - 15504*a[5]^16 - 38760*a[5]^15 - 77520*a[5]^14 -
    125970*a[5]^13 - 167960*a[5]^12 - 184756*a[5]^11 - 167960*a[5]^10 -
    125970*a[5]^9 - 77520*a[5]^8 - 38760*a[5]^7 - 15504*a[5]^6 - 4845*a[5]^5
a[3]^5*a[4]^27 - a[5]^22 - 21*a[5]^21 - 210*a[5]^20 - 1330*a[5]^19 -
    5985*a[5]^18 - 20349*a[5]^17 - 54264*a[5]^16 - 116280*a[5]^15 -
    203490*a[5]^14 - 293930*a[5]^13 - 352716*a[5]^12 - 352716*a[5]^11 -
    293930*a[5]^10 - 203490*a[5]^9 - 116280*a[5]^8 - 54264*a[5]^7 -
    20349*a[5]^6 - 5985*a[5]^5 - 1330*a[5]^4 - 210*a[5]^3 - 21*a[5]^2 -
    a[5],
a[3]^4*a[4]^29 - a[5]^23 - 22*a[5]^22 - 231*a[5]^21 - 1540*a[5]^20 -
    7315*a[5]^19 - 26334*a[5]^18 - 74613*a[5]^17 - 170544*a[5]^16 -
    319770*a[5]^15 - 497420*a[5]^14 - 646646*a[5]^13 - 705432*a[5]^12 -
    646646*a[5]^11 - 497420*a[5]^10 - 319770*a[5]^9 - 170544*a[5]^8 -
    74613*a[5]^7 - 26334*a[5]^6 - 7315*a[5]^5 - 1540*a[5]^4 - 231*a[5]^3 -
    22*a[5]^2 - a[5],
a[3]^3*a[4]^31 - a[5]^24 - 23*a[5]^23 - 253*a[5]^22 - 1771*a[5]^21 -
    8855*a[5]^20 - 33649*a[5]^19 - 100947*a[5]^18 - 245157*a[5]^17 -
    490314*a[5]^16 - 817190*a[5]^15 - 1144066*a[5]^14 - 1352078*a[5]^13 -
    1352078*a[5]^12 - 1144066*a[5]^11 - 817190*a[5]^10 - 490314*a[5]^9 -
    245157*a[5]^8 - 100947*a[5]^7 - 33649*a[5]^6 - 8855*a[5]^5 - 1771*a[5]^4
a[3]^2*a[4]^33 - a[5]^25 - 24*a[5]^24 - 276*a[5]^23 - 2024*a[5]^22 -
    10626*a[5]^21 - 42504*a[5]^20 - 134596*a[5]^19 - 346104*a[5]^18 -
    735471*a[5]^17 - 1307504*a[5]^16 - 1961256*a[5]^15 - 2496144*a[5]^14 -
    2704156*a[5]^13 - 2496144*a[5]^12 - 1961256*a[5]^11 - 1307504*a[5]^10 -
    735471*a[5]^9 - 346104*a[5]^8 - 134596*a[5]^7 - 42504*a[5]^6 -
    10626*a[5]^5 - 2024*a[5]^4 - 276*a[5]^3 - 24*a[5]^2 - a[5],
a[3]*a[4]^35 - a[5]^26 - 25*a[5]^25 - 300*a[5]^24 - 2300*a[5]^23 -
    12650*a[5]^22 - 53130*a[5]^21 - 177100*a[5]^20 - 480700*a[5]^19 -
    1081575*a[5]^18 - 2042975*a[5]^17 - 3268760*a[5]^16 - 4457400*a[5]^15 -
```

```
        5200300*a[5]^14 - 5200300*a[5]^13 - 4457400*a[5]^12 - 3268760*a[5]^11 -
        2042975*a[5]^10 - 1081575*a[5]^9 - 480700*a[5]^8 - 177100*a[5]^7 -
        53130*a[5]^6 - 12650*a[5]^5 - 2300*a[5]^4 - 300*a[5]^3 - 25*a[5]^2 -
        a[5],
    a[3]*a[5] + a[3] - a[4]^2,
    a[4]^37 - a[5]^27 - 26*a[5]^26 - 325*a[5]^25 - 2600*a[5]^24 - 14950*a[5]^23
        - 65780*a[5]^22 - 230230*a[5]^21 - 657800*a[5]^20 - 1562275*a[5]^19 -
        3124550*a[5]^18 - 5311735*a[5]^17 - 7726160*a[5]^16 - 9657700*a[5]^15 -
        10400600*a[5]^14 - 9657700*a[5]^13 - 7726160*a[5]^12 - 5311735*a[5]^11 -        3
        65780*a[5]^6 - 14950*a[5]^5 - 2600*a[5]^4 - 325*a[5]^3 - 26*a[5]^2 -
        a[5]
]
0.630
[
    f37^2 - f11*f7^9 + f37,
    f37*f27 - f11^2*f7^6,
    f27^2 - f11^3*f7^3 - f17,
    f37*f17 - f11^3*f7^3,
    f37*f11 - f27*f7^3 + f11,
    f27*f17 - f37*f7 - f7,
    f11^4 - f37*f7,
    f27*f11 - f17*f7^3,
    f17^2 - f27*f7,
    f17*f11 - f7^4
]
0.050
```

I did get integralClosure in Macaulay2 version 1.2 to finally give me:

```
i9 : toString(P#0)

o9 = QQ[w_6, w_7, w_1, u]/(w_1^7*u^3+w_1^3+u^7,w_7*w_1^2*u^3-829440*w_1^8*u^2+
     4148928*w_1^6*u^19+1016064*w_1^5*u^9-282475249*w_1^2*u^53-69177612*w_1*u^43     -248
     2343938939766144*w_1^6*u^61-16862305517568*w_1^6*u^24+39081926349850776*w_1     ^5*u
     14356626006067632*w_1^4*u^78-80330149896192*w_1^4*u^41-2036393312256*w_1^4*     u^4+
     39081926349850776*w_1*u^85+134720355555072*w_1*u^48-1179869773824*w_1*u^11+     1435
```

Since I have no idea how to extract the ideal from this, given that the presentation command didn't work for some reason, I used MAGMA

```
Q:=Rationals();
P<w_6, w_7, w_1, u>:=PolynomialRing(Q,4);
e1:=w_1^7*u^3+w_1^3+u^7;
e2:=w_7*w_1^2*u^3-829440*w_1^8*u^2+4148928*w_1^6*u^19+1016064*w_1^5*u^9-282475249*w_1^2*u
e3:=w_6*u^2+282475249*w_1^6*u^50+69177612*w_1^5*u^40-580608*w_1^5*u^3+16941456*w_1^4*u^3
e4:=w_6*w_1-w_7*u^2;
e5:=w_6*w_7-687970713600*w_1^11+79792266297612001*w_1^6*u^98-2343938939766144*w_1^6*u^61
```

```
e6:=w_6^2+282475249*w_6*w_1^6*u^48-16941456*w_7*w_1^6*u^23+69177612*w_7*w_1^4*u^40-580608


I:=ideal<P|e1>;
N:=function(g) l:=NormalForm(g,I); return l; end function;

//reducuing w_7
NormalForm(N(e2*(w_1^5*u^3+w_1))+w_7*u^10,ideal<P|u^10>) div 82944;
//7*w_1^6*u^9 - 3*w_1^2*u^6/u^10

//reducing w_6
NormalForm(N(-(7*w_1^6*u^3-3*w_1^2)*(w_1^6*u^3+w_1^2)),ideal<P|u^9>) div -3;
//w_1*u^7/u^9
```

to see that (with $a := w_1$) this can be reduced to

$b := (7 * a^6 * u^3 - 3 * a^2)/u^4$ and $c := a/u^2$, with $a^7 * u^3 + a^3 + u^7 = 0$. And from there, $(b + 3 * c^2)/7 = a^6/u = c^6 * u^1 1$ and $a = c * u^2$; so b and a are not necessary, as advertised.

And SINGULAR on

```
> ring r=13,(w,y,u),lp;
> ideal i=y7+y3u-u11,wu+y2;
> list nor=normalP(i,"withRing");
> nor;
[1]:
   [1]:
//   characteristic : 13
//   number of vars : 2
//        block   1 : ordering dp
//                  : names    T(3)
//        block   2 : ordering lp
//                  : names    u
//        block   3 : ordering C
   [2]:
//   characteristic : 13
//   number of vars : 1
//        block   1 : ordering lp
//                  : names    w
//        block   2 : ordering C
[2]:
   [1]:
      _[1]=yu5
      _[2]=y2
      _[3]=yu3
      _[4]=u6
   [2]:
```

```
      _[1]=1
[3]:
   [1]:
      8,0
   [2]:
      10
> def R=nor[1][1];
> setring R;
> normap;
normap[1]=-T(3)^2*u^5
normap[2]=T(3)*u^3
normap[3]=u
> norid;
norid[1]=T(3)^15*u^33+T(3)^8*u^23+3*T(3)^4*u^12+T(3)^3-2*T(3)*u^13-u
norid[2]=T(3)^7*u^15+T(3)^3*u^4-u^5
norid[3]=T(3)^7*u^14+T(3)^3*u^3-u^4
norid[4]=T(3)^8*u^16+T(3)^4*u^5-T(3)*u^6
norid[5]=2*T(3)^8*u^25+T(3)^7*u^13+2*T(3)^4*u^14+T(3)^3*u^2-2*T(3)*u^15-u^3
norid[6]=T(3)^24*u^55+3*T(3)^6*u^24+4*T(3)^5*u^12+T(3)^4-T(3)^3*u^25-3*T(3)^2*u^13-T(3)*
norid[7]=T(3)^7*u^21+T(3)^3*u^10-u^11
norid[8]=T(3)^7*u^20+T(3)^3*u^9-u^10
norid[9]=T(3)^7*u^19+T(3)^3*u^8-u^9
norid[10]=T(3)^8*u^21+T(3)^4*u^10-T(3)*u^11
> option(redSB);
> ideal j=std(norid);j;
j[1]=T(3)^7*u^11+T(3)^3-u
```

works fairly well, though it slows down considerably, even by $q = 23$.