

# Annotated Examples 1

## Disguised Hermitian curve

The Hermitian curve in one-point form (used for coding over  $\mathbf{F}_{16}$  because it has 64 other rational points there) is usually given in a form such as  $y^4 + y = x^5$ , which emphasizes its *trace-norm form*, since the left-hand side is the *trace* of  $y$  from  $\mathbf{F}_{16}$  to  $\mathbf{F}_4$  and the right hand side is the *norm*  $x^4 \cdot x$  of  $x$  from  $\mathbf{F}_{16}$  to  $\mathbf{F}_4$ . At the one-point  $P_\infty$ , the rational functions  $x$  and  $y$  have pole orders 4 and 5 respectively; so rename them  $f_4$  and  $f_5$ , and use a grevlex-over-weight monomial order to get the curve

$$\overline{\mathbf{F}}_2[f_5; f_4] / \langle f_5^4 + f_5 - f_4^5 \rangle.$$

As an  $\overline{\mathbf{F}}_2[f_4]$ -module, this has a *basis* ( $f_0 := 1, f_5, f_{10} := f_5^2, f_{15} := f_5^3$ ); so as an algebra over  $\overline{\mathbf{F}}_2[f_4]$ , it has a presentation as

$$\overline{\mathbf{F}}_2[f_{15}, f_{10}, f_5; f_4] / \langle I \rangle$$

with  $I$  having a Gröbner basis

$$\begin{aligned} f_{15}^2 - (-f_{15} + f_{10}f_4^5) \\ f_{15}f_{10} - (-f_{10} + f_5f_4^5) \\ f_{10}^2 - (-f_5 + f_4^5) \\ f_{15}f_5 - (-f_5 + f_4^5) \\ f_{10}f_5 - f_{15} \\ f_5^2 - f_{10} \end{aligned}$$

which describes the (non-trivial part of the) multiplication of module basis elements.

Now disguise this using  $f_9 := f_5f_4$  and  $f_8 := f_4^2$  and arbitrary characteristic to get

$$\mathbf{F}[f_9; f_8] / \langle f_9^8 - f_8^9 + 2f_9f_8^6 - f_9^2f_8^3 \rangle$$

I used this example to show that the normal function in SINGULAR 3-0-4 and the integralClosure function in Macaulay2 version 1.1 both terminated prematurely. (Also it shows that MAGMA's IntegralClosure function uses a much too restrictive separability condition.)

In SINGULAR 3-1-0, now

```

                                SINGULAR
A Computer Algebra System for Polynomial Computations / version 3-1-0
                                0<
                                \ Mar 2009
                                \
    by: G.-M. Greuel, G. Pfister, H. Schoenemann
FB Mathematik der Universitaet, D-67653 Kaiserslautern \
> LIB "normal.lib";
> intmat A[2][2]=1,0,9,8;
> ring r=0,(f9,f8),M(A);
> ideal i=f9^8-f8^9+2*f9*f8^6-f9^2*f8^3;
> list nor=normal(i);
> nor;
[1]:
    [1]:
// characteristic : 0
// number of vars : 7
//      block 1 : ordering dp
//              : names T(1) T(2) T(3) T(4) T(5)
//      block 2 : ordering M
//              : names f9 f8
//              : weights 1 0
//              : weights 9 8
//      block 3 : ordering C
[2]:
    [1]:
    _[1]=f9^5*f8^2
    _[2]=f9^6*f8
    _[3]=-f9^4*f8^2+f9^3*f8^5
    _[4]=f9^8
    _[5]=-f9^2*f8^5+f9*f8^8
    _[6]=-f9^2*f8^3+f9*f8^6
> def R=nor[1][1];
> setring R;
> normap;
normap[1]=f9
normap[2]=f8
> norid;
norid[1]=T(1)*f9-T(2)*f8
norid[2]=T(4)*f9-T(5)*f8+f9
norid[3]=-T(5)+f8^2
norid[4]=-T(3)*f8+f9^2
norid[5]=T(2)*f9^2-T(4)*f8
norid[6]=T(1)*f8^3-T(2)*f8-T(3)*f9^2
norid[7]=T(3)*f8^3-T(5)*f9^2
norid[8]=-T(2)*f9*f8+T(2)*f8^4-T(3)*f9^3
```

```

norid[9]=T(3)*f9^5-T(4)*f8^5-T(5)*f9+T(5)*f8^3
norid[10]=T(4)*f8^8-T(5)*f9^7+T(5)*f9*f8^3-T(5)*f8^6
norid[11]=T(1)^2-f8
norid[12]=T(1)*T(2)-f9
norid[13]=T(2)^2-T(3)
norid[14]=T(1)*T(3)-T(2)*f9
norid[15]=T(2)*T(3)-T(4)
norid[16]=T(3)^2-T(1)*f8^2+T(2)
norid[17]=T(1)*T(4)-T(3)*f9
norid[18]=T(2)*T(4)-T(1)*f8^2+T(2)
norid[19]=T(3)*T(4)+T(3)-f9*f8^2
norid[20]=T(4)^2-T(1)*f9^2*f8+T(4)
norid[21]=T(1)*T(5)-T(1)*f8^2
norid[22]=T(2)*T(5)-T(1)*f9*f8
norid[23]=T(3)*T(5)-f9^2*f8
norid[24]=T(4)*T(5)-T(1)*f9^3
norid[25]=T(5)^2-f8^4
norid[26]=-f9^8+f9^2*f8^3-2*f9*f8^6+f8^9

```

At least,

```

> list l=interred(norid);l;
[1]:
_[1]=T(5)-f8^2
_[2]=T(4)*f8^8-T(5)*f9^7+T(5)*f9*f8^3-T(5)*f8^6
_[3]=T(4)*f9-T(5)*f8+f9
_[4]=T(3)*f8-f9^2
_[5]=T(3)*f9^5-T(4)*f8^5-T(5)*f9+T(5)*f8^3
_[6]=T(2)*f9*f8-T(2)*f8^4+T(3)*f9^3
_[7]=T(2)*f9^2-T(4)*f8
_[8]=T(1)*f8^3-T(2)*f8-T(3)*f9^2
_[9]=T(1)*f9-T(2)*f8
_[10]=T(4)^2-T(1)*f9^2*f8+T(4)
_[11]=T(3)*T(4)+T(3)-f9*f8^2
_[12]=T(2)*T(4)-T(1)*f8^2+T(2)
_[13]=T(1)*T(4)-T(3)*f9
_[14]=T(3)^2-T(1)*f8^2+T(2)
_[15]=T(2)*T(3)-T(4)
_[16]=T(1)*T(3)-T(2)*f9
_[17]=T(2)^2-T(3)
_[18]=T(1)*T(2)-f9
_[19]=T(1)^2-f8
_[20]=f9^8-f9^2*f8^3+2*f9*f8^6-f8^9

```

makes it easier to see that  $T(5)$  is redundant by any standards, and that  $T(4), T(3), T(2)$ , and  $T(1)$ , should have weights 15, 10, 5, and 4 respectively.

So I would choose to edit this as follows, to get a  $\mathbf{Q}[f_4]$  algebra presentation with module basis  $(f_0 := 1, f_5, f_{10}, f_{15})$ .

```
>intmat B[4][4]=
  1, 1,1,0,
  1, 1,0,0,
  1, 0,0,0,
  15,10,5,4;
> ring ic0=0, (f15,f10,f5,f4),M(B);
> map phi=R0,f4,f5,f10,f15,f4^4,f5*f4,f4^2;
> ideal j0=std(phi(norid));j0;
j0[1]=f5^2-f10
j0[2]=f10*f5-f15
j0[3]=f15*f5+f5-f4^5
j0[4]=f10^2+f5-f4^5
j0[5]=f15*f10+f10-f5*f4^5
j0[6]=f15^2+f15-f10*f4^5
```

This should be almost independent of characteristic, so try this in some relatively small generic (not 2,3,5) characteristic, say 23, using both normal and normalP:

```

> ring r23=23,(f9,f8),M(A);
> ideal i23=f9^8=f8^9+2*f9*f8^6-f9^2*f8^3;
> list nor23=normal(i23);
> nor23;[1]:[1]:
//   characteristic : 23
//   number of vars : 7
//       block   1 : ordering dp
//               : names   T(1) T(2) T(3) T(4) T(5)
//       block   2 : ordering M
//               : names   f9 f8
//               : weights 0 1
//               : weights 9 8
//       block   3 : ordering C
[2]:[1]:
  _[1]=f9^5*f8^2
  _[2]=f9^6*f8
  _[3]=f9^3*f8^5-f9^4*f8^2
  _[4]=f9^8
  _[5]=f9*f8^8-f9^2*f8^5
  _[6]=f9*f8^6-f9^2*f8^3
> def R23=nor23[1][1];
> setring R23;
> normap;
normap[1]=f9
normap[2]=f8
> norid;
norid[1]=T(1)*f9-T(2)*f8
norid[2]=T(4)*f9-T(5)*f8+f9
norid[3]=-T(5)+f8^2
norid[4]=-T(3)*f8+f9^2
norid[5]=T(2)*f9^2-T(4)*f8
norid[6]=T(1)*f8^3-T(2)*f8-T(3)*f9^2
norid[7]=T(3)*f8^3-T(5)*f9^2
norid[8]=T(2)*f8^4-T(2)*f9*f8-T(3)*f9^3
norid[9]=T(3)*f9^5-T(4)*f8^5+T(5)*f8^3-T(5)*f9
norid[10]=T(4)*f8^8-T(5)*f8^6+T(5)*f9*f8^3-T(5)*f9^7
norid[11]=T(1)^2-f8
norid[12]=T(1)*T(2)-f9
norid[13]=T(2)^2-T(3)
norid[14]=T(1)*T(3)-T(2)*f9
norid[15]=T(2)*T(3)-T(4)
norid[16]=T(3)^2-T(1)*f8^2+T(2)

```

```

norid[17]=T(1)*T(4)-T(3)*f9
norid[18]=T(2)*T(4)-T(1)*f8^2+T(2)
norid[19]=T(3)*T(4)+T(3)-f9*f8^2
norid[20]=T(4)^2-T(1)*f9^2*f8+T(4)
norid[21]=T(1)*T(5)-T(1)*f8^2
norid[22]=T(2)*T(5)-T(1)*f9*f8
norid[23]=T(3)*T(5)-f9^2*f8
norid[24]=T(4)*T(5)-T(1)*f9^3
norid[25]=T(5)^2-f8^4
norid[26]=f8^9-2*f9*f8^6+f9^2*f8^3-f9^8
> list l=interred(norid);l;
[1]:
  _[1]=T(5)-f8^2
  _[2]=T(4)*f8^8-T(5)*f9^7+T(5)*f9*f8^3-T(5)*f8^6
  _[3]=T(4)*f9-T(5)*f8+f9
  _[4]=T(3)*f8-f9^2
  _[5]=T(3)*f9^5-T(4)*f8^5-T(5)*f9+T(5)*f8^3
  _[6]=T(2)*f9*f8-T(2)*f8^4+T(3)*f9^3
  _[7]=T(2)*f9^2-T(4)*f8
  _[8]=T(1)*f8^3-T(2)*f8-T(3)*f9^2
  _[9]=T(1)*f9-T(2)*f8
  _[10]=T(4)^2-T(1)*f9^2*f8+T(4)
  _[11]=T(3)*T(4)+T(3)-f9*f8^2
  _[12]=T(2)*T(4)-T(1)*f8^2+T(2)
  _[13]=T(1)*T(4)-T(3)*f9
  _[14]=T(3)^2-T(1)*f8^2+T(2)
  _[15]=T(2)*T(3)-T(4)
  _[16]=T(1)*T(3)-T(2)*f9
  _[17]=T(2)^2-T(3)
  _[18]=T(1)*T(2)-f9
  _[19]=T(1)^2-f8
  _[20]=f9^8-f9^2*f8^3+2*f9*f8^6-f8^9

```

The new normalP function in SINGULAR 3-1-0, together with the "withring" option:

```

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A Computer Algebra System for Polynomial Computations / version 3-1-0
                                0<
by: G.-M. Greuel, G. Pfister, H. Schoenemann \ Mar 2009
FB Mathematik der Universitaet, D-67653 Kaiserslautern \
> LIB "normal.lib";
> ring r=23,(y,x),wp(9,8);
> ideal i=y8-y2x3+2yx6-x9;
> list norp=normalP(i,"withring");
> norp;

```

```

[1]:
  [1]:
// characteristic : 23
// number of vars : 2
//      block   1 : ordering dp
//            : names   T(1) T(3)
//      block   2 : ordering C
[2]:
  [1]:
    _[1]=y5x
    _[2]=y6
    _[3]=y3x4-y4x
    _[4]=x8-y2x2
    _[5]=yx5-y2x2
[3]:
  [1]:
    22
  [2]:
    22
> def R=norpp[1][1];
> setring R;
> normap;
normap[1]=T(1)^6-T(1)*T(3)^2
normap[2]=T(1)^2
> norid;
norid[1]=T(1)^17-3*T(1)^12*T(3)^2+3*T(1)^7*T(3)^4-T(1)^7*T(3)-T(1)^2*T(3)^6+T(1)^2*T(3)^
norid[2]=T(1)^11*T(3)-2*T(1)^6*T(3)^3+T(1)*T(3)^5-T(1)*T(3)^2
norid[3]=T(1)^10-2*T(1)^5*T(3)^2+T(3)^4-T(3)
norid[4]=T(1)^10*T(3)-2*T(1)^5*T(3)^3+T(3)^5-T(3)^2
norid[5]=T(1)^15-3*T(1)^10*T(3)^2+3*T(1)^5*T(3)^4-T(1)^5*T(3)-T(3)^6+T(3)^3
norid[6]=T(1)^12-2*T(1)^7*T(3)^2+T(1)^2*T(3)^4-T(1)^2*T(3)
norid[7]=T(1)^12*T(3)-2*T(1)^7*T(3)^3+T(1)^2*T(3)^5-T(1)^2*T(3)^2
norid[8]=T(1)^11-2*T(1)^6*T(3)^2+T(1)*T(3)^4-T(1)*T(3)
norid[9]=T(1)^48-8*T(1)^43*T(3)^2+5*T(1)^38*T(3)^4-10*T(1)^33*T(3)^6+T(1)^28*T(3)^8-10*T
> option(redSB);
> ideal j=std(norid);j;
j[1]=T(1)^10-2*T(1)^5*T(3)^2+T(3)^4-T(3)

```

So its answer is a grevlex one on  $T(1)$  and  $T(3)$ , of implicit weights 4 and 10.

Using the slightly newer normalP option "withRing" with or without the additional "noRed" option:

```

                                SINGULAR /
A Computer Algebra System for Polynomial Computations / version 3-1-0
                                0<
by: G.-M. Greuel, G. Pfister, H. Schoenemann \ Mar 2009

```

```

FB Mathematik der Universitaet, D-67653 Kaiserslautern \
> LIB "presolve.lib";
> LIB "normal.lib";
> ring r=23,(y,x),wp(9,8);
> ideal i=y8-y2x3+2yx6-x9;
> list norp=normalP(i,"withRing","noRed");
> norp;
[1]:
  [1]:
// characteristic : 23
// number of vars : 6
//   block   1 : ordering dp
//             : names   T(1) T(2) T(3) T(4)
//   block   2 : ordering wp
//             : names   y x
//             : weights 9 8
//   block   3 : ordering C
[2]:
  [1]:
    _[1]=y5x
    _[2]=y6
    _[3]=y3x4-y4x
    _[4]=x8-y2x2
    _[5]=yx5-y2x2
[3]:
  [1]:
    22
  [2]:
    22
> def R=norp[1][1];
> setring R;
> normap;
normap[1]=y
normap[2]=x
> norid;
norid[1]=T(1)*y-T(2)*x
norid[2]=-T(3)*x+y^2
norid[3]=T(2)*y^2-T(4)*x+2*x
norid[4]=-T(4)*y+x^3+y
norid[5]=T(1)*x^3-T(1)*y-T(3)*y^2
norid[6]=T(1)^2-x
norid[7]=T(1)*T(2)-y
norid[8]=T(2)^2-T(3)
norid[9]=T(1)*T(3)-T(2)*y
norid[10]=T(2)*T(3)-T(4)+2
norid[11]=T(3)^2-T(1)*x^2+T(2)

```

```

norid[12]=T(1)*T(4)-2*T(1)-T(3)*y
norid[13]=T(2)*T(4)-T(1)*x^2-T(2)
norid[14]=T(3)*T(4)-T(3)-y*x^2
norid[15]=T(4)^2-T(2)*y*x^2-3*T(4)+2
norid[16]=y^8-x^9+2*y*x^6-y^2*x^3
> option(redSB);
> ideal j=std(norid);j;
j[1]=y^8-x^9+2*y*x^6-y^2*x^3
j[2]=T(4)*y-x^3-y
j[3]=T(4)*x^6-y^7-3*x^6+y*x^3
j[4]=T(3)*x-y^2
j[5]=T(3)*y^5-T(4)*x^5+3*x^5-y*x^2
j[6]=T(2)*y^2-T(4)*x+2*x
j[7]=T(2)*x^4-T(2)*y*x-T(3)*y^3
j[8]=T(1)*y-T(2)*x
j[9]=T(1)*x^3-T(2)*x-T(3)*y^2
j[10]=T(4)^2-T(2)*y*x^2-3*T(4)+2
j[11]=T(3)*T(4)-T(3)-y*x^2
j[12]=T(2)*T(4)-T(1)*x^2-T(2)
j[13]=T(1)*T(4)-2*T(1)-T(3)*y
j[14]=T(3)^2-T(1)*x^2+T(2)
j[15]=T(2)*T(3)-T(4)+2
j[16]=T(1)*T(3)-T(2)*y
j[17]=T(2)^2-T(3)
j[18]=T(1)*T(2)-y
j[19]=T(1)^2-x

```

which gives an answer grevlex in 4 new variables  $T(1)$ ,  $T(2)$ ,  $T(3)$ ,  $T(4)$  of implicit weights 4,5,10,15, over the original ring with variables  $y$  and  $x$  of weights 9 and 8, without realizing that the variables  $y$  and  $x$  are no longer of any use.

For Macaulay2's integralClosure function (March 15, 2009 version ?), the following in characteristic 23 which used to take me about 57 seconds now seems to take between one and two second:

```

Macaulay 2, version 1.2
with packages: Elimination, IntegralClosure, LLLBases, PrimaryDecomposition,
               ReesAlgebra, SchurRings, TangentCone

i1 : load "IntegralClosure.m2";
i2 : R=ZZ/23[y,x,MonomialOrder=>{Weights=>{9,8}}];
i3 : I=ideal(y^8-y^2*x^3+2*y*x^6-x^9);
i4 : S=R/I;
i5 : time P=presentation(integralClosure(S))
    -- used 1.34 seconds

o5 = | -w_(12,0)^6w_(7,0)+w_(12,0)^2w_(10,0)^3+w_(10,0)^2w_(7,0)^2
      -w_(12,0)^6w_(10,0)+w_(10,0)^3w_(7,0)+w_(7,0)
      w_(12,0)^5-w_(10,0)^4-w_(10,0)
      w_(12,0)^3w_(10,0)^2-w_(12,0)^2w_(7,0)
      -w_(12,0)^6+w_(12,0)w_(10,0)+w_(10,0)^2w_(7,0) |

i6 : time G=gens gb P
    -- used 0. seconds

o6 = | w_(10,0)^14+w_(10,0)^11-w_(7,0)^5
      w_(12,0)w_(7,0)^4-w_(10,0)^12-w_(10,0)^9
      w_(12,0)w_(10,0)^2-w_(7,0)
      w_(12,0)^2w_(7,0)^3-w_(10,0)^10-w_(10,0)^7
      w_(12,0)^3w_(7,0)^2-w_(10,0)^8-w_(10,0)^5
      w_(12,0)^4w_(7,0)-w_(10,0)^6-w_(10,0)^3 w_(12,0)^5-w_(10,0)^4-w_(10,0) |

```

The speed is vastly improved, but the output is in terms of functions that would have weights 4,5, and 14! I don't see any theoretical reasoning that would justify answers other than either 4,5 or 4,5,10,15. M2 seems to try to produce a minimal number of generators, but misses here despite one of the relations being

$$w_{(12,0)}w_{(10,0)}^2-w_{(7,0)}$$

which shows that  $f_{14} = f_5^2 f_4$ .

The charactersitic 0 version is:

```

i8 : R=QQ[y,x,MonomialOrder=>{Weights=>{9,8}}]
i9 : I=ideal(y^8-y^2*x^3+2*y*x^6-x^9)
i10 : S=R/I
i11 : time P=presentation(integralClosure(S))
      -- used 1.54 seconds

o11 = | -w_(12,0)^6w_(7,0)+w_(12,0)^2w_(10,0)^3+w_(10,0)^2w_(7,0)^2
      -w_(12,0)^6w_(10,0)+w_(10,0)^3w_(7,0)+w_(7,0)
      w_(12,0)^5-w_(10,0)^4-w_(10,0)
      w_(12,0)^3w_(10,0)^2-w_(12,0)^2w_(7,0)
      -w_(12,0)^6+w_(12,0)w_(10,0)+w_(10,0)^2w_(7,0) |

i12 : time G=gens gb P
      -- used 0. seconds

o12 = | w_(10,0)^14+w_(10,0)^11-w_(7,0)^5
      w_(12,0)w_(7,0)^4-w_(10,0)^12-w_(10,0)^9
      w_(12,0)w_(10,0)^2-w_(7,0)
      w_(12,0)^2w_(7,0)^3-w_(10,0)^10-w_(10,0)^7
      w_(12,0)^3w_(7,0)^2-w_(10,0)^8-w_(10,0)^5
      w_(12,0)^4w_(7,0)-w_(10,0)^6-w_(10,0)^3
      w_(12,0)^5-w_(10,0)^4-w_(10,0) |

```

which is consistent with the char 23 result. This is vastly superior to the segmentation faults and silly output that used to be produced.

Let's try MAGMA. There is no way to use weights; and we know it would complain about separability in char 2.

```

t:=Cputime();
Q:=Rationals();
F<x>:=FunctionField(Q);
P<y>:=PolynomialRing(F);
f:=y^8-y^2*x^3+2*y*x^6-x^9;
Ff<Y>:=RationalExtensionRepresentation(FunctionField(f));
C<X>:=CoefficientRing(Ff);
INT:=Integers(C);
IC:=IntegralClosure(INT,Ff);
B:=Basis(IC);
for i in [1..#B] do
i,B[i];
end for;
"time for char=0 is",Cputime(t);
-----
1 1
2 Y
3 1/X*Y^2
4 1/X*Y^3
5 1/X^2*Y^4
6 1/X^2*Y^5
7 1/X^3*Y^6
8 1/X^13*Y^7 + 1/X^10*Y^6 + 1/X^7*Y^5 + 1/X^4*Y^4 - 1/X^10*Y + 1/X^7
time for char=0 is 0.080

t23:=Cputime();
Q:=GF(23);
F<x>:=FunctionField(Q);
P<y>:=PolynomialRing(F);
f:=y^8-y^2*x^3+2*y*x^6-x^9;
Ff<Y>:=RationalExtensionRepresentation(FunctionField(f));
C<X>:=CoefficientRing(Ff);
INT:=Integers(C);
IC:=IntegralClosure(INT,Ff);
B:=Basis(IC);
for i in [1..#B] do
i,B[i];
end for;
"time for char=23 is",Cputime(t23);
-----
1 1
2 Y
3 1/X*Y^2

```

```

4 1/X*Y^3
5 1/X^2*Y^4
6 1/X^2*Y^5
7 1/X^3*Y^6
8 1/X^13*Y^7 + 1/X^10*Y^6 + 1/X^7*Y^5 + 1/X^4*Y^4 + 22/X^10*Y + 1/X^7
time for char=23 is 0.090

```

In both cases, it gives an  $\mathbf{F}[f_8]$ -module basis with elements of weights 0, 9, 10, 19, 20, 29, 30, 4, again obscuring the function of weight 5. And the integral closure is treated a subset of, maybe,  $\mathbf{F}(X)[Y]/\langle f \rangle$ , with no explicit version given.

Try MAGMA's Normalisation function instead.

```

t:=Cputime();
SetLogFile("may2609");
F:=Rationals();
P<y,x>:=PolynomialRing(F,2,"weight",[1,0,9,8]);
f:=y^8-y^2*x^3+2*y*x^6-x^9;
I:=ideal<P|f>;
N:=Normalisation(I);
J:=N[1][1];J;
"Normalisation time=",Cputime(t);
G:=GroebnerBasis(J);G;
"total time=",Cputime(t);

```

The output ideal (in a ring with lex!!! monomial ordering) is generated by:

```

-J.1^21+J.1^16*J.2^4+J.1^11*J.2^5+J.1^6*J.2^6+J.1^6+J.1*J.2^7-J.1*J.2,
-J.1^7+J.1^2*J.2^4+J.1^2*J.2,
-J.1^10+2*J.1^5*J.2+J.2^8-J.2^2,
-J.1^11+J.1^6*J.2^4+J.1*J.2^5+J.1*J.2^2,
-J.1^12+2*J.1^7*J.2+J.1^2*J.2^8-J.1^2*J.2^2,
-J.1^11*J.2+2*J.1^6*J.2^2+J.1*J.2^9-J.1*J.2^3,
-J.1^17+J.1^12*J.2^4+J.1^7*J.2^5+J.1^2*J.2^6+J.1^2*J.2^3,
-J.1^10*J.2^2+2*J.1^5*J.2^3+J.2^10-J.2^4,
-J.1^6+J.1*J.2^4+J.1*J.2,
-J.1^12*J.2^2+2*J.1^7*J.2^3+J.1^2*J.2^10-J.1^2*J.2^4,
-J.1^11*J.2^3+2*J.1^6*J.2^4+J.1*J.2^11-J.1*J.2^5,
-J.1^7*J.2+J.1^2*J.2^5+J.1^2*J.2^2,
-J.1^10*J.2^4+2*J.1^5*J.2^5+J.2^12-J.2^6,
-J.1^6*J.2^2+J.1*J.2^6+J.1*J.2^3,
-J.1^5+J.2^4+J.2

```

Normalisation time= 0.840 seconds

which then has Gröbner basis:

```
J.1^5-J.2^4-J.2
```

total time= 0.840



```

WT_MATRIX_T= [
  [ 19, 15, 14, 10, 9, 5, 4, 8 ]
]
time for q= 13 is 0.040 seconds
modulus= 1001
[
  f_9^3*f_8^12,
  f_9^7*f_8^7 - f_9*f_8^10,
  f_9^6*f_8^8 + f_9^7*f_8^5 + f_8^11 - f_9*f_8^8,
  f_9^2*f_8^12,
  f_9*f_8^13,
  f_9^5*f_8^8 + f_9^6*f_8^5 + f_9^7*f_8^2 + f_8^8 - f_9*f_8^5,
  f_9^4*f_8^9 + f_9^5*f_8^6 + f_9^6*f_8^3 + f_9^7 + f_8^6 - f_9*f_8^3,
  f_8^14,
  f_8^13
]
1 f_9^3*f_8^12
2 f_9^7*f_8^7 - f_9*f_8^10
3 f_9^6*f_8^8 + f_9^7*f_8^5 + f_8^11 - f_9*f_8^8
4 f_9^2*f_8^12
5 f_9*f_8^13
6 f_9^5*f_8^8 + f_9^6*f_8^5 + f_9^7*f_8^2 + f_8^8 - f_9*f_8^5
7 f_9^4*f_8^9 + f_9^5*f_8^6 + f_9^6*f_8^3 + f_9^7 + f_8^6 - f_9*f_8^3
8 f_8^14
9 f_8^13
newrelations= [
  f_4^2 - f_8,
  f_5^2 - f_10,
  f_5*f_4 - f_9,
  f_9^2 - f_10*f_8,
  f_9*f_5 - f_14,
  f_9*f_4 - f_5*f_8,
  f_10^2 - f_4*f_8^2 + f_5,
  f_10*f_9 - f_19,
  f_10*f_5 - f_15 - 1,
  f_10*f_4 - f_14,
  f_14^2 - f_4*f_8^3 + f_5*f_8,
  f_14*f_10 - f_8^3 + f_9,
  f_14*f_9 - f_15*f_8 - f_8,
  f_14*f_5 - f_19,
  f_14*f_4 - f_10*f_8,
  f_15^2 - f_14*f_8^2 + 3*f_15 + 2,
  f_15*f_14 - f_5*f_8^3 + 2*f_14,
  f_15*f_10 - f_9*f_8^2 + 2*f_10,
  f_15*f_9 - f_8^3 + 2*f_9,
  f_15*f_5 - f_4*f_8^2 + 2*f_5,

```

```

f_15*f_4 - f_19 + f_4,
f_19^2 - f_14*f_8^3 + f_15*f_8 + f_8,
f_19*f_15 - f_10*f_8^3 + 2*f_19,
f_19*f_14 - f_9*f_8^3 + f_10*f_8,
f_19*f_10 - f_5*f_8^3 + f_14,
f_19*f_9 - f_4*f_8^3 + f_5*f_8,
f_19*f_5 - f_8^3 + f_9,
f_19*f_4 - f_15*f_8 - f_8
]
totaltime= 0.160 seconds

in terms of a  $\mathbf{Q}[f_8]$  module basis  $f_0 := 1, f_9, f_{10}, f_{19}, f_4, f_5, f_{14}, f_{15}$ .
Minimization gives:

> load "closure_interchange.mag";

[15,10,5,4]
[
v_15^2+3*v_15-v_10*v_4^5+2,
v_15*v_10+2*v_10-v_5*v_4^5,
v_10^2+v_5-v_4^5,
v_15*v_5+2*v_5-v_4^5,
v_10*v_5-v_15-1,
v_5^2-v_10
]
time5= 0.210

```

Note that 5 was considered a bad prime in the input, whereas 3 is also a bad prime in that the integral closure is clearly larger than the reduction mod 3 would imply.